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Locally Factorial Generic Zariski Surfaces Are Factorial

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INTRODUCTION

In this article we study normal affine sufaces in A_k^3 defined by equations of the form $z^p = G(x, y)$, where the ground field k is assumed to be algebraically closed of characteristic $p \neq 0$. If G is such that G_x, G_y, and $G_{xx}G_{yy} - G_{xy}^2$ have no points in common then such surfaces are called Zariski surfaces (a title given to them by P. Blass in [B1]). If we also assume that $z^p = G$ has the maximum possible number of singularities ($(\deg G - 1)^2$ if p does not divide $\deg G$ or $(\deg G)^2 - 3(\deg G) + 3$ otherwise) then we call such a surface a generic Zaraski surface because $z^p = G$ is of this type for a generic choice of G. In this paper we develop a technique for calculating the divisor class group of Zariski surfaces at a singular point. We then employ this technique to show that locally factorial generic Zariski surfaces are factorial.

NOTATION

1. k—algebraically closed field of characteristic $p \neq 0$.
2. A_k^3—affine 3-space over k.

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3. Surface—irreducible, reduced, two-dimensional, quasi-projective variety over k.

4. The notation $F: f(x, y, z) = 0$ means $F = \mathrm{Spec}(k[x, y, z]/(f(x, y, z)))$.

5. If A is a Krull ring we denote by $\mathrm{Cl}(A)$ the divisor class group of A.

6. If F is a surface in A_k^3 then $\mathrm{Cl}(F)$ denotes the divisor class group of the coordinate ring of F.

7. For $f \in k[x, y, z]$ we denote by

   $\deg f$ the total degree of f,
   $\deg_x f$ the degree of f in x,
   $\deg_y f$ the degree of f in y.

1. Preliminaries

The following results, (1.1) to (1.6), can be found in P. Samuel's 1964 Tata notes [S1]. For the definition of a Krull ring and the divisor class group of a Krull ring the reader is referred to Samuel's notes or to R. Fossum's book "The Divisor Class Group of a Krull Domain" (see [F]). The rings that are studied in this article are coordinate rings or localizations of coordinate rings of normal affine surfaces (hence are noetherian integrally closed domains) and are thus Krull rings.

Notation. Let $A \subset B$ be rings. Let $p \subset A$ and $q \subset B$ be prime ideals. We write q/p if $q \cap A = p$ and we say that q lies over p.

THEOREM 1.1. *Let $A \subset B$ be Krull rings. Suppose that either B is integral over A or B is A-flat. Then there is a well-defined group homomorphism $\varphi: \mathrm{Cl}(A) \to \mathrm{Cl}(B)$ (see [S1, pp. 19–20]).*

The homomorphism of Theorem 1.1 is defined in the following manner. If q and p are height one primes of B and A with q/p we let $e(q:p)$ be the ramification index of q over p. Then for each height one prime p of A we define $\varphi([p]) = \sum_{q/p} e(q:p)[q]$ to be the sum taken over all height one primes in B lying over p. The sum is always finite since B is a Krull ring. We then extend φ. Let B be a Krull ring of characteristic $p \neq 0$ and E be the quotient field of B. Let Δ be a derivation of E such that $\Delta(B) \subset B$. Let $K = \mathrm{Ker}\, \Delta$ and $A = B \cap K$. Then A is a Krull ring with B integral over A. Thus we have a map $\varphi: \mathrm{Cl}(A) \to \mathrm{Cl}(B)$. Set $L = \{t^{-1}\Delta t\colon t \in E \text{ and } t^{-1}\Delta t \in B\}$. Set $L^1 = \{u^{-1}\Delta u\colon u \text{ is a unit in } B\}$. Then L^1 is a subgroup of L.

THEOREM 1.2. (a) *There exists a canonical monomorphism* $\bar{\varphi}$: ker $\varphi \to L/L^1$. (b) *If* $[E; K] = p$ *and* $\Delta(B)$ *is not contained in any height one prime of* B, *then* $\bar{\varphi}$ *is an isomorphism* [S1, p. 62].

THEOREM 1.3. (a) *If* $[E:K] = p$, *then there exists an* $a \in A$ *such that* $\Delta^p = a\Delta$ *and* (b) *an element* $t \in B$ *is in* L *if and only if* $\Delta^{p-1} - at = -t^p$ [S1, pp. 63–64].

Remark 1.4. $\bar{\varphi}$: ker $\varphi \to L/L^1$ is described as follows. If $\beta \in \ker \varphi$, then $\varphi(\beta) = tB$ for some $t \in E$. $\bar{\varphi}$ takes β to $t^{-1}\Delta t$.

THEOREM 1.5. *Let* A *be a Krull ring and* S *a multiplicatively closed subset in* A. *Then* $S^{-1}A$ *is an* A*-flat Krull ring and* $\varnothing$: Cl $A \to$ Cl $S^{-1}A$ *is surjective, where the* ker $\varnothing$ *is generated by the divisor classes of height one primes that intersect* S [S1, p. 21].

THEOREM 1.6. *Let* A *be a noetherian ring and* m *an ideal contained in the Jacobson radical of* A. *Let* A *have the* m*-adic topology and* $\hat{A}$ *be the completion of* A. *Then if* $\hat{A}$ *is a Krull ring, so is* A. *Also* $\varnothing$: $\mathrm{Cl}(A) \to \mathrm{Cl}(\hat{A})$ *is an injection* [S1, p. 23].

Remark 1.7. By the Jacobian criterion F: $z^p = G(x, y)$ is normal if and only if G satisfies the condition that G_x and G_y have no common factors. Hereafter we will always assume that G satisfies this condition (see [M, p. 125]).

DEFINITION 1.8. Let D: $k(x, y) \to k(x, y)$ be the k-derivation defined by $D = G_y(\partial/\partial x) - G_x(\partial/\partial y)$. D is called the *Jacobian derivation of* G.

LEMMA 1.9. *Let* L *be the group of logarithmic derivatives of* D *in* $k[x, y]$. *Then if* $t \in L$, *then* $\deg t \leqslant \deg G - 2$ [L4, p. 394].

THEOREM 1.10. *Let* $A = k[x^p, y^p, G]$ *and* ϑ *be the coordinate ring of* F: $z^p = G$. *Then* (a) $A = D^{-1}(0) \cap k[x, y]$, (b) $\vartheta \simeq A$, *and* (c) $\mathrm{Cl}(A) \simeq L = \{f^{-1}Df\colon f \in k(x, y)$ *and* $f^{-1}Df \in k[x, y]\}$, *the group of logarithmic derivatives of* D *in* $k[x, y]$ [L4, pp. 393–394].

THEOREM 1.11. *Let* D *be the Jacobian derivation of* G. *Then* $\forall \alpha \in k(x, y)$, $D^{p-1}\alpha - a\alpha = -\sum_{i=0}^{p-1} G^i\, \nabla(G^{p-i-1})$, *where* $\nabla = \partial^{2p-2}/(\partial x^{p-1}\, \partial y^{p-1})$, *and* $D^p = aD$ [L4, p. 395].

THEOREM 1.12. *Let* Q *be a singularity of the surface* F: $z^p = G$. *Then* $\mathrm{Cl}(\vartheta_Q) \simeq L/L_Q$, *where* L *is the group of logarithmic derivatives of* D *in* $k[x, y]$ *and* $L_Q = \{f^{-1}Df\colon f^{-1}Df \in k[x, y], f \in k(x, y), f$ *is defined at* Q *and* $f(Q) \neq 0\}$.

Proof. By (1.5) the sequence $0 \to \ker \varnothing \to \mathrm{Cl}(A) \to \mathrm{Cl}(S^{-1}A) \to 0$ is exact. Let $(\mathbf{p})$ be the divisor class of a height one prime $\mathbf{p}$ such that $\mathbf{p} \cap S \neq \varnothing$, where S is the complement of the maximal ideal M_Q of $A = k[x^p, y^p, G]$ corresponding to the singularity Q of F. Let $f \in k[x, y]$ generate the unique height one prime of $k[x, y]$ lying over $\mathbf{p}$. $\mathbf{p} \cap S \neq \varnothing$ implies that $f(Q) \neq 0$. By (1.4), $f^{-1}Df \in L_Q$. Therefore the restriction of the isomorphism $\mathrm{Cl}(A) \to^{\simeq} L$ of (1.10) to $\ker \varnothing$ is an injection of $\ker \varnothing$ into L_Q. It is easy to see that the restriction is surjective as well. Thus we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L_Q \arrow[r] & L & & \\
0 \arrow[r] & \ker \varnothing \arrow[r] \arrow[u, "\cong"] & \mathrm{Cl}(A) \arrow[r] \arrow[u, "\cong"] & \mathrm{Cl}(S^{-1}A) \arrow[r] & 0.
\end{tikzcd}$$

We conclude that $\mathrm{Cl}(\vartheta_Q) \simeq \mathrm{Cl}(S^{-1}A) \simeq L/L_Q$.

Remark 1.13. P. Blass in [B1] showed that if G_x, G_y, and $W = G_{xx}G_{yy} - G_{xy}^2$ have no points in common, the $F\colon z^P = G$ has only isolated double point singularities with local equation of the form $z^P = xy + (\text{higher-degree terms})$. Hereafter we will refer to this condition on G as condition (B). It can also be shown that for a generic G, the surface $F\colon z^p = G$ has $(g-1)^2$ distinct singularities if p does not divide $\deg G$ and $g^2 - 3g + 3$ distinct singularities if p divides $\deg G$. When we assume that both of these conditions are satisfied (i.e., (B) and F has the maximum possible number of singularities) we will call F a *generic Zariski surface.*

THEOREM 1.14. $\mathrm{Cl}(\vartheta_Q) \simeq 0$ *or* $\mathbb{Z}/p\mathbb{Z}$ *if* G *satisfies condition* (B) (*see* [13, p. 632]).

Remark 1.15. In [L1] Lang described a technique for computing the divisor class group of ϑ. We now extend this method to provide a computational technique for determining $\mathrm{Cl}(\vartheta_Q)$. First, we determine L, the group of logarithmic derivatives of D in $k[x, y]$. Then we evaluate each of these at Q and if we obtain only zeroes then $\mathrm{Cl}(\vartheta_Q) = 0$ otherwise $\mathrm{Cl}(\vartheta_Q) \simeq \mathbb{Z}/p\mathbb{Z}$. We prove this result in several steps.

STEP 1: LEMMA 1.16. *Let* G *satisfy condition* (B), D *be the Jacobian derivation of* G, *and* $a = -\sum_{i=0}^{p-1} G^i \nabla(G^{p-i-1})$, *where* $\nabla = \partial^{2p-2}/(\partial x^{p-1} \partial y^{p-1})$. *Then* (a) $D^P = aD$ *and* (b) $a(Q) \neq 0$ *at each singularity* Q *of* F.

Proof. In (1.11) if we let $\alpha = 1$ we obtain the desired formula for a.

To proof (b) we can assume after a linear change of coordinates that $Q = (0, 0)$ and F has the form $z^p = xy + (\text{higher-degree terms})$ by

(1.11). Then $a(Q)=a(0,0)= -\nabla(G^{p-1})\quad(0,0)= -(1+(\text{higher-degree terms}))(0,0)= -1$.

DEFINITION 1.17. For each singularity Q of F, let H_Q be the additive subgroup of k consisting of the roots of the polynomial $h(t)=t^p-a(Q)\,t$. Note that $H_Q \simeq \mathbb{Z}/p\mathbb{Z}$ by (1.16).

STEP 2: LEMMA 1.18. *For each singularity Q of F, $t\in L$ implies $t(Q)\in H_Q$.*

Proof. If Q is a singularity of F and $h\in k[x,y]$, then $(Dh)(Q)=h_x(Q)\,G_y(Q)-h_y(Q)\,G_x(Q)=0$. If $t\in L$, $D^{p-1}t-at=-t^p$ by (1.11). If we evaluate this expression at Q we have $a(Q)\,t(Q)=(t(Q))^p$. Therefore $t(Q)\in H_Q$.

THEOREM 1.19. *Let G satisfy condition* (B). *Then* $\mathrm{Cl}(\vartheta_Q)\cong\mathbb{Z}/p\mathbb{Z}$ *if and only if the map L to H_Q of* (1.18) *is not the zero mapping.*

Proof. Again we may assume the $Q=(0,0)$ and $z^p=xy+(\text{higher-degree terms})$. We have the following commutative diagram by (1.5), (1.6), (1.10), and (1.12), where $\tilde{L}=\{Df/f\in k[[x,y]]\colon f\in k(x,y)\}$ and $\tilde{L}^1=\{Df/f\colon f \text{ is a unit in } k[[x,y]]\}$:

$$\begin{array}{ccccc}
\mathrm{Cl}(\vartheta) & \xrightarrow{\text{surjection}} & \mathrm{Cl}(\vartheta_Q) & \xrightarrow{\text{injection}} & \mathrm{Cl}(\hat{\vartheta}_Q)\simeq\mathbb{Z}/p\mathbb{Z} \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
L & \xrightarrow{\text{surjection}} & L/L_Q & \xrightarrow{\text{injection}} & \tilde{L}/\tilde{L}^1\simeq\mathbb{Z}/p\mathbb{Z}.
\end{array}$$

In [L3, p. 632] we showed that $\mathrm{Cl}(\hat{\vartheta}_Q)\simeq Z/pZ$. In $k[[x,y]]$, G can be factored into $G=uv$, where $u=x+(\text{higher-degree terms})$ and $v=y+(\text{higher-degree terms})$. We also showed in [L4, p. 632] that Du/u generates $\tilde{L}/\tilde{L}^1$.

$$\frac{Du}{u}=\frac{u_xG_y-u_yG_x}{u}=\frac{x+(\text{higher-degree terms})_1}{x+(\text{higher-degree terms})_2}.$$

This implies that $Du/u=1+(\text{higher-degree terms})$ in $k[[x,y]]$.

($\Rightarrow$) If $\mathrm{Cl}(\vartheta_Q)\cong\mathbb{Z}/p\mathbb{Z}$ then from the commutative diagram we have that $L/L_Q\simeq\tilde{L}/\tilde{L}^1$. Therefore there is a $t\in L$ such that $Du/u-t\in\tilde{L}^1$. If h is a unit in $k[[x,y]]$, then $(Dh/h)\ (0,0)=0$.

Therefore $(Du/u)\ (0,0)-t(0,0)=0$. This implies that $t(0,0)=1$ and the map from L to H_Q is not trivial.

($\Leftarrow$) If $t\in L$ and $t(0,0)\neq 0$ then $t\notin L_Q$. Therefore $\mathrm{Cl}(\vartheta_Q)=\mathbb{Z}/p\mathbb{Z}$. This proves the theorem.

THEOREM 1.20. *Let G satisfy condition (B). If F is locally factorial then F is factorial.*

Proof (by contradiction). Suppose F is not factorial. Then by (1.10) there is a nonzero $t \in L$. Let $t = S_1 S_2 \cdots S_n$ be a factorization of t into prime factors. Each S_i is relatively prime to either G_x or G_y since G_x and G_y have no common factors. We may assume that $S_1, S_2, ..., S_r$ are prime to G_x and that $S_{r+1}, S_{r+2}, ..., S_n$ are prime to G_y. Then for each $i = 1, 2, ..., r$, S_i and G_x intersect in at most $\deg(S_i) \cdot (g-1)$ distinct points, where $g = \deg G$, by Bezout's theorem. For $i = r+1, ..., n$, S_i and G_y have at most $\deg(S_i) \cdot (g-1)$ points of intersection. The total number of these intersection points is at most $\sum_{i=1}^{n} (g-1) \deg S_i = (g-1) \deg t \leqslant (g-1)(g-2)$ by (1.9). Since G_x and G_y intersect in at least $g^2 - 3g + 3$ distinct points (see (1.13)), we can choose Q, a singularity of F that does not satisfy any of the S_i's. Then $t(Q) \neq 0$. By (1.19), $\mathrm{Cl}(\vartheta_Q) \neq 0$ and F is not locally factorial.

In the proof of (1.20) we proved the following facts.

LEMMA 1.21. *If $t \in L$, then t is not zero in at least $(g-1)^2 - (\deg t)(g-1)$ singularities if g is not divisible by p and $(g^2 - 3g + 3) - (\deg t)(g-1)$ singularities if g is divisible by p.*

THEOREM 1.22. *Let $n(F)$ denote the number of singularities of F. The map $L \to \bigoplus_{Q \in \mathrm{Sing}(F)} H_Q$ defined by $t \to (t(Q_1), t(Q_2), ..., t(Q_{n(F)}))$ is an injection.*

EXAMPLE 1.23. The surface $Z^p = xy + x^{p+1} + y^{p+1}$ was studied in [L3] and shown to have class group of order p^{2p-1}. Since $t = 1 \in L$, we have that $\mathrm{Cl}(\vartheta_Q) \cong \mathbb{Z}/p\mathbb{Z}$ at each singularity [L3, p. 401].

EXAMPLE 1.24. The surface $Z^5 = xy + x^3 + y^3 + x^2y^2$ has class group of order p^3. L is generated by $1 + xy$, $x + y$, and $x + \theta y$, where θ is a primitive ninth root of unity. Since these three logarithmic derivatives have no points in common, $\mathrm{Cl}(\vartheta_Q) = \mathbb{Z}/p\mathbb{Z}$ at each singularity.

Remark 1.25. We have not as yet been able to find an example of a generic Zariski surface F: $z^p = G$ that has nontrivial class group but is factorial at a singularity.

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